A Learning Framework for Blind Source Separation Using Generalized Eigenvalues*

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Abstract. This paper presents a learning framework for blind source separation (BSS), in which the BSS is formulated as generalized Eigenvalue (GE) problem. Compared to the typical information-theoretical approaches, this new one has at least two merits: (1) the unknown unmixing matrix directly works out from the GE equation without time-consuming iterative learning; (2) The correctness of the solution is guaranteed. We give out a general learning procedure under this framework. The computer simulation shows validity of our method.

1 Introduction

At present, many authors engage in blind source separation (BSS) or independent component analysis (ICA) research work, and a lot of studying literature (for a review, see [1]) have been published. The most basic form of BSS can be stated as follows: Suppose there are n channels of source signals with at most one Gaussian source signal, denoted as $s_1(t), s_2(t), ..., s_n(t)$, which are statistically independent each other. The sources are instantaneously and linearly mixed by an unknown full-rank square matrix $\mathbf{A}$ and observed as:

$$\mathbf{x(t)} = \mathbf{As(t)}, \tag{1}$$

where $\mathbf{s(t)} = [s_1(t), s_2(t), ..., s_n(t)]^T, \mathbf{x(t)} = [x_1(t), x_2(t), ..., x_n(t)]^T$, and T is a transpose operation of a matrix. The objective of an ICA approach is to recover $s(t)_i's$ up to a constant scale and any permutation of indices through a set of observations $\mathbf{x(t)}$ by finding out a de-mixing matrix $\mathbf{W}$ such that

$$\mathbf{y(t)} = \mathbf{Wx(t)}, \tag{2}$$

where $\mathbf{y(t)} = [y_1(t), y_2(t), ..., y_n(t)]^T$ is a recovered signal of $\mathbf{s(t)}$.

In the literature, one approach initiated from the seminal work of Stone [6] is to formulate the BSS as generalized Eigenvalue (GE) problem. Compared to the

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typical information-theoretic based approaches, e.g., INFOMAX [2], negentropy [3], cumulant [4], and ML [5], the GE-based ones have at least two merits: (1) the unknown un-mixing matrix directly works out from the GE equation without time-consuming iterative learning; (2) The correctness of the solution can be guaranteed.

In Stone's paper [6], a metric named *Temporal Predictability* has been presented as a logarithm of a ratio of two prediction error terms. The numerator is the summation of long-term prediction errors of a $\boldsymbol{y}$'s component, while the numerator is the summation of its short-term prediction errors. Essentially, his work is based on the conjecture that, *given any set of statistically independent source signals, the temporal predictability of any signal mixture is less than (or equal to) that of any of its component source signals*. Unfortunately, although a number of experiments have reported its success, some empirical studies have found that this conjecture is not totally correct, as pointed out in our recent paper [7]. Under the circumstances, we have proposed a new metric called Independency Metric, through which a new BSS algorithm with global convergence is presented [7].

In this paper, we further present a general BSS learning framework formulated as GE problems. We have given out a generalized contrast function, whereby a general BSS learning procedure is obtained with those algorithms in [7, 8] as its particular examples. We have analyzed the global convergence property of such a learning, and shown that it guarantees to acquire a correct BSS solution.

2 Contrast Function Extracting Source Signals

In instantaneous linear mixture model Eq.(1), suppose there exist two operators g and h, for any real number k_i and k_j, satisfying the following relationship:

$$g(k_i s_i(t) + k_j s_j(t)) = k_i^2 g(s_i(t)) + k_j^2 g(s_j(t)), \forall i, j \in \{1, 2, ..., n\}, i \neq j. \tag{3}$$

$$h(k_i s_i(t) + k_j s_j(t)) = k_i^2 h(s_i(t)) + k_j^2 h(s_j(t)), \forall i, j \in \{1, 2, ..., n\}, i \neq j. \tag{4}$$

$$g(k_i s_i(t), k_j s_j(t)) = 0, h(k_i s_i(t), k_j s_j(t)) = 0, \forall i, j \in \{1, 2, ..., n\}, i \neq j. \tag{5}$$

Hence, we have

$$g(\mathbf{x}(t)) = \mathbf{A} \cdot diag([g(s_1(t)), g(s_2(t)), ..., g(s_n(t))])\mathbf{A}^T. \tag{6}$$

For the recovered signal $\mathbf{y}(t)$ given by Eq.(2), we can then obtain:

$$g(\mathbf{y}(t)) = \mathbf{WA} \cdot diag([g(s_1(t)), g(s_2(t)), ..., g(s_n(t))])(\mathbf{WA})^T. \tag{7}$$

We know that each component y of $\mathbf{y}$ is a linear mixture of n sources with:

$$y = \mathbf{wx}, \tag{8}$$

where **w** is a n-dimensional row vector. We then define a general form of contrast function of BSS making use of generalized eigenvalue as

$$L(\mathbf{wx}) = \frac{g(\mathbf{wx})}{h(\mathbf{wx})} = \frac{\mathbf{w}g(\mathbf{x})\mathbf{w}^T}{\mathbf{w}h(\mathbf{x})\mathbf{w}^T}, \tag{9}$$

For the new redundancy reduction metric, similar to the proof in [7], we can get the following theorem:

Theorem 1. For the source signals $s_1, s_2, ..., s_n$, suppose there exist two operators g and h satisfying Eq.(4) and Eq.(5) so that $L(s_i)$'s are not equal each other, i.e.,

$$\frac{g(s_i)}{h(s_i)} \neq \frac{g(s_j)}{h(s_j)}, i \neq j.$$

Denote

$$L(s_{i_0}) = max\{L(s_1), L(s_2), ..., L(s_n)\}. \tag{10}$$

For any mixing signal y described in Eq.(8), we then have

$$L(y) \leq L(s_{i_0}). \tag{11}$$

If and only if $y = ks_{i_0}$, where k is any non-zero real number, then

$$L(y) = L(s_{i_0}). \tag{12}$$

In the above theorem, operators h and g may have various forms. For instance, $h(s(t)) = var(s(t)), g(s(t)) = var(\int_0^t s(\tau)d\tau)$ (used in [7]); $h(s(t)) = E(s^2(t))$, $g(s(t)) = \sum_{i=0}^t E(s^2(i))$ (used in [8]).

3 Globally Optimal Analysis of BSS Algorithm

3.1 Equivalent Form About Gradient of Contrast Function

Since $L(s_j)$'s are not equal each other, without loss of generality, we assume that

$$L(s_1) > L(s_2) > ... > L(s_n). \tag{13}$$

According to Theorem 1 and Eq.(9), we therefore have

$$Q(\mathbf{w}) = logL(\mathbf{wx}) \leq logL(s_1). \tag{14}$$

This implies that the source signal s_1 can be extracted through solving following optimization problem:

$$\max_{\mathbf{w} \neq 0} Q(\mathbf{w}). \tag{15}$$

The objective function in Eq.(15) can transform

$$Q(\mathbf{w}) = logL(\mathbf{wx}) = log\frac{\mathbf{w}g(\mathbf{x})\mathbf{w}^T}{\mathbf{w}h(\mathbf{x})\mathbf{w}^T}. \tag{16}$$

We let $\|\mathbf{z}\|_2$ be a norm of vector $\mathbf{z}=(z_1, z_2, ..., z_n)$, and $\|\mathbf{z}\|_2=\sqrt{z_1^2+z_2^2+...+z_n^2}$. In Eq.(16),

$$Q(\mathbf{w}) = \log((\frac{\mathbf{w}}{\|\mathbf{w}\|_2})g(\mathbf{x})(\frac{\mathbf{w}}{\|\mathbf{w}\|_2})^T)/((\frac{\mathbf{w}}{\|\mathbf{w}\|_2})h(\mathbf{x})(\frac{\mathbf{w}}{\|\mathbf{w}\|_2})^T), \tag{17}$$

Therefore, from Eq.(16) and (17), we obtain

$$\max_{\mathbf{w}\neq 0} Q(\mathbf{w}) = \max_{\|\mathbf{w}\|_2=1} Q(\mathbf{w}) \tag{18}$$

Since $h(\mathbf{x}) = \mathbf{A}h(\mathbf{s})\mathbf{A}^T, g(\mathbf{x}) = \mathbf{A}g(\mathbf{s})\mathbf{A}^T$, $\mathbf{A}$ is a nonsingular square matrix, and both h($\mathbf{s}$) and g($\mathbf{s}$) are two diagonal matrices, we know that h($\mathbf{x}$) and g($\mathbf{x}$) are real symmetrical and positive definite matrices. Therefore, Q($\mathbf{w}$) is a logarithm of ratio of two positive definite quadratic forms, and it is continuous and differentiable. Since the set $\{\mathbf{w}|\|\mathbf{w}\|_2 = 1\}$ is a closed set, according to Eq.(18), we know that the objective function Q($\mathbf{w}$) in (15) exists global maximum and global minimum. Since Q($\mathbf{w}$) is differentiable, all optimal solutions in (15) must be stable point.

With some mathematical computations, we can finally obtain the gradient of Q($\mathbf{w}$):

$$\begin{aligned}\bigtriangledown Q(\mathbf{w}) &= \frac{2g(\mathbf{x})\mathbf{w}^T}{\mathbf{w}g(\mathbf{x})\mathbf{w}^T} - \frac{2h(\mathbf{x})\mathbf{w}^T}{\mathbf{w}h(\mathbf{x})\mathbf{w}^T} = \frac{2}{\mathbf{w}g(\mathbf{x})\mathbf{w}^T}\{g(\mathbf{x})\mathbf{w} - \frac{\mathbf{w}g(x)\mathbf{w}^T}{\mathbf{w}h(\mathbf{x})\mathbf{w}^T}h(\mathbf{x})\mathbf{w}^T\} \\ &= \frac{2}{\mathbf{w}g(\mathbf{x})\mathbf{w}^T}\{g(\mathbf{x})\mathbf{w}^T - L(\mathbf{wx})h(\mathbf{x})\mathbf{w}^T\}.\end{aligned} \tag{19}$$

If and only if $\bigtriangledown Q(\mathbf{w}) = 0$, we obtain

$$g(\mathbf{x})\mathbf{w}^T = L(\mathbf{wx})h(\mathbf{x})\mathbf{w}^T. \tag{20}$$

3.2 Generalized Eigenvalue Problem

Definition 1. Suppose that $\mathbf{A}$ is an $n \times n$ real symmetrical matrix and $\mathbf{B}$ is an $n \times n$ real symmetrical and positive definite matrix, the following eigenvalue problem:

$$\mathbf{Ax} = \lambda\mathbf{Bx} \tag{21}$$

is called generalized eigenvalue problem; the number λ satisfying Eq.(20) is called eigenvalue of matrix A relative to matrix B; the nonzero solution relative to λ is called eigenvector belonging to λ.

Note that both $g(\mathbf{x})$ and $h(\mathbf{x})$ are real symmetrical and positive definite matrices. Therefore, Eq.(20) is a generalized eigenvalue problem, where $L(\mathbf{wx})$ is an eigenvalue about the problem and $\mathbf{w}^T$ is an eigenvector corresponding to eigenvalue $L(\mathbf{wx})$. Furthermore, solving Eq.(15) actually becomes a problem solving generalized eigenvalue vector. Suppose $\mathbf{w}_0$ is a stable point of Q($\mathbf{w}$) in (15) and is also an eigenvector corresponding to eigenvalue $L(\mathbf{w}_0^T\mathbf{x})$ in Eq.(20). From

Eq.(16), we then have $Q(\mathbf{w}_0) = logL(\mathbf{w}_0\mathbf{x})$. If $L(\mathbf{w}_0\mathbf{x})$ is the largest eigenvalue in Eq.(20), $\mathbf{w}_0$ must be the global optimal solution in Eq.(15). Thus, through solving an eigenvector corresponding to the largest eigenvalue in Eq.(20), we can recover the source signals.

3.3 Method of Recovering All Source Signals

Theorem 2. Under the condition of Theorem 1, $\hat{\mathbf{w}}x$ is a recovered signal of source signals if and only if it is a stable point of optimization problem Eq.(15).

Proof: Sufficiency of the condition. Suppose that $\hat{\mathbf{w}}$ is a stable point of optimization problem Eq.(15). From Eq.(20), we obtain

$$g(\mathbf{x})\hat{\mathbf{w}}^T = L(\hat{\mathbf{w}}\mathbf{x})h(\mathbf{x})\hat{\mathbf{w}}^T. \tag{22}$$

Let $\mathbf{w}_i = (\overbrace{0, \ldots, 0, 1}^{i}, 0, \ldots, 0)\mathbf{A}^{-1}$(i=1,2,. . . ,n), then all eigenvalues are $L(\mathbf{w}_1\mathbf{x})$, $L(\mathbf{w}_2\mathbf{x}), ..., L(\mathbf{w}_n\mathbf{x})$, in generalized eigenvalue problem Eq.(20). assume $L(\hat{\mathbf{w}}x) = L(\mathbf{w}_{i_1}\mathbf{x})$ where $1 \le i_1 \le n$, and $\hat{\mathbf{w}}$ is an eigenvector corresponding to eigenvalue $L(\mathbf{w}_{i_1}\mathbf{x})$. Because the linear uncorrelated eigenvector corresponding to eigenvalue is exclusive in Eq.(20) and $\mathbf{w}_{i_1}$ is a eigenvector corresponding to eigenvalue $L(\mathbf{w}_{i_1}\mathbf{x})$, we have $\hat{\mathbf{w}} = c_1\mathbf{w}_{i_1}$, in which c_1 is a non-zero constant. Hence, we obtain $\hat{\mathbf{w}\mathbf{x}} = (c_1\mathbf{w}_{i_1})\mathbf{x} = c_1 s_{i_1}$.

Necessity of the condition. If $\hat{\mathbf{w}}\mathbf{x}$ is a recovered signal of source signals, there exists a source signal s_{i_2} $(1 \le i_2 \le n)$ such that $\hat{\mathbf{w}}\mathbf{x} = c_2 s_{i_2}$, in which c_2 is non-zero constant. Similar to the proof of Eq.(36), from Eq.(19), we have

$$g(\mathbf{x})\hat{\mathbf{w}} = L(\hat{\mathbf{w}}\mathbf{x})h(\mathbf{x})\hat{\mathbf{w}} \tag{23}$$

According to Eq.(19) and Eq.(20), $\hat{\mathbf{w}}$ is a stable point of optimization problem Eq.(15). □

Corollary 1. $\hat{\mathbf{w}}_i x$ is a recovered signal of source signal s_i if and only if $\hat{\mathbf{w}}_i$ is an eigenvector corresponding to the eigenvalue in Eq.(20).

Proof: On the basis of above proof and Eq.(19), it can be seen that this result is true. □

According to Theorem 2, de-mixing matrix W can be obtained through solving eigenvector corresponding to eigenvalue $L(s_1), L(s_2), ..., L(s_n)$ in Eq.(20).

4 Simulation Results

In our computer simulation, we let h(s) be signal s and the operator g be,

$$g(\mathbf{s}(t)) = \begin{cases} var(\sum_{i=1}^{t} \mathbf{s}(i)), t \le q; \\ var(\sum_{i=t-q+1}^{t} \mathbf{s}(i)), t > q. \end{cases} \tag{24}$$

We used three independent source signals: a subgaussian signal s_1(a cosine signal), a supergaussian signal s_2(a speech sound), and a gaussian signal s_3 generated using the randn procedure in Matlab. In this example, mixing matrix **A**

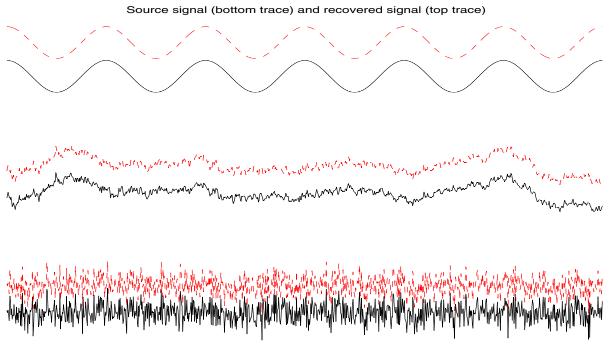

Fig. 1. Separating Mixture signals of Super-Guassian Signal, Sub-Gaussian Signal and Guassian Signal

was generated at random and the number of samples was 5,000. Each source signal (solid line) and its corresponding recovered signal (dot line) acquired by our proposed algorithm are shown in Figure 1.

5 Conclusion

In this paper, we have presented a general BSS learning procedure using general Eigenvalues. Such a learning not only acquires the BSS solution in the one step without the time-consuming iterative learning as used in those information-theoretic based algorithms, but also makes a correct BSS solution guaranteed. The computer simulations have shown the success of our method.

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